An extension of the characterization of the domain of attraction of an asymptotically stable fixed point in the case of a nonlinear discrete dynamical system

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Abstract

The purpose of this paper is to show that if the spectral radius of the matrix of the linearized system of a nonlinear discrete dynamical system is less then one then the characterization theorem from [2, 3] holds.

1 Introduction

We consider the system of difference equations

$$x_{k+1} = g(x_k) \qquad k = 0, 1, 2... \tag{1}$$

where $g : \Omega \to \Omega$ is an analytic function defined on a domain Ω included in $\mathbb{R}^n$.

A point $x^0 \in \Omega$ is a fixed point for the system (1) if x^0 satisfies

$$x^0 = g(x^0) \tag{2}$$

The fixed point x^0 of (1) is "stable" provided that given any ball $B(x^0, \varepsilon) = \{x \in \Omega / \|x - x^0\| < \varepsilon\}$, there is a ball $B(x^0, \delta) = \{x \in \Omega / \|x - x^0\| < \delta\}$ such that if $x \in B(x^0, \delta)$ then $g^k(x) \in B(x^0, \varepsilon)$, for $k = 0, 1, 2, ...$ [4].

If in addition there is a ball $B(x^0, r)$ such that $g^k(x) \to x^0$ as $k \to \infty$ for all $x \in B(x^0, r)$ then the fixed point x^0 is "asymptotically stable".[4].

The domain of attraction $DA(x^0)$ of the asymptotically stable fixed point x^0 is the set of initial states $x \in \Omega$ from which the system converges to the fixed point itself i.e.

$$DA(x^0) = \{x \in \Omega | g^k(x) \xrightarrow{k\to\infty} x^0\} \tag{3}$$

It is known that x^0 is a fixed point for system (1) if and only if $0 \in \mathbb{R}^n$ is a fixed point for the system

$$y_{k+1} = f(y_k) \qquad k = 0, 1, 2... \tag{4}$$

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where $f : \Omega - x^0 \to \Omega - x^0$ is the analytic function defined by

$$f(y) = g(y + x^0) - x^0 \qquad \text{for } y \in \Omega - x^0 \tag{5}$$

The fixed point x^0 of (1) is asymptotically stable if and only if the fixed point $0 \in \mathbb{R}^n$ of the system (4) is asymptotically stable.

The domain of attraction of x^0, $DA(x^0)$ is related to the domain of attraction of 0, $DA(0)$ by the equation

$$DA(x^0) = DA(0) + x^0 \tag{6}$$

For the above reason in the followings instead of the system (1) we will consider the system (4).

Theoretical research shows that the $DA(0)$ and its boundary are complicated sets. In most cases, they do not admit an explicit elementary representation. For this reason, different procedures are used for the approximation of the $DA(0)$ with domains having a simpler shape. For example, in the case of the theorem 4.20 pg. 170 [4] the domain which approximates the $DA(0)$ is defined by a Lyapunov function V built with the matrix $\partial_0 f$ of the linearized system in 0. In [3], we presented a technique for the construction of a Lyapunov function V in the case when the matrix $\partial_0 f$ is a contraction, i.e. $\|\partial_0 f\| < 1$. In this paper, we extend this result for the more general case when $r(\partial_0 f) < 1$ (where r denotes the spectral radius). The Lyapunov function V is built using the whole nonlinear system, not only the matrix $\partial_0 f$. V is defined on the whole $DA(0)$, and more, the $DA(0)$ is the natural domain of analyticity of V.

2 The result

Let be $f : \Omega \to \Omega$ an analytic function defined on a domain $\Omega \subset \mathbb{R}^n$ containing the origin $0 \in \mathbb{R}^n$.

Theorem 1. *If the function f satisfies the following conditions:*

$$f(0) = 0 \tag{7}$$

$$r(\partial_0 f) < 1 \tag{8}$$

then 0 is an asymptotically stable fixed point. $DA(0)$ is an open subset of Ω and coincides with the natural domain of analyticity of the unique solution V of the iterative first order functional equation

$$\begin{cases} V(f(x)) - V(x) = -\|x\|^2 \\ V(0) = 0 \end{cases} \tag{9}$$

The function V is positive on $DA(0)$ and $V(x) \overset{x \to x^0}{\longrightarrow} +\infty$, for any $x^0 \in FrDA(0)$ ($FrDA(0)$ denotes the boundary of $DA(0)$).

Proof. Let be $A = \partial_0 f$ and $r = r(\partial_0 f)$.

The fact 0 is an asymptotically stable fixed point is proved in [1]. The fact that $DA(0)$ is an open subset of Ω follows from the continuity of the function f^k, for any $k \in \mathbb{N}$.

We will prove that the series $\sum\limits_{k=0}^{\infty} \|f^k(x)\|^2$ is convergent for any $x \in DA(0)$.

We can decompose the function f as follows:

$$f(x) = Ax + h(x) \tag{10}$$

where $h : \Omega \to \Omega$ is an analytical function which satisfies $h(0) = 0$.

As $r < 1$, there exists $c > 0$ such that

$$\|A^k x\| < cr^k \|x\| \qquad \forall x \in \Omega, k \in \mathbb{N} \tag{11}$$

Let be $\bar{c} = \max\{1, c\}$ and $\varepsilon = \frac{1-r}{2\bar{c}} > 0$. As h is continuous and $h(0) = 0$, it follows that there exists $\delta > 0$ such that

$$\|h(x)\| < \varepsilon \|x\| \qquad \forall x \in B(0, \delta). \tag{12}$$

Let be $x \in DA(0)$ and $x_k = f^k(x)$. This provides the existence of $k_x \in \mathbb{N}$ such that $x_k \in B(0, \delta)$, for any $k \geq k_x$. Let be $y_k = x_{k+k_x}$. This sequence satisfies $y_k \in B(0, \delta)$ for any $k \in \mathbb{N}$ and it also satisfies (4).

The formula of variation of constants gives:

$$y_k = A^k y_0 + \sum_{i=l}^{k-1} A^{k-i-1} h(y_i) \qquad \forall k \in \mathbb{N}^\star \tag{13}$$

Relations (11) and (13) provide

$$\|y_k\| \leq cr^k \|y_0\| + \sum_{i=0}^{k-1} cr^{k-i-1} \|h(y_i)\| \qquad \forall k \in \mathbb{N}^\star \tag{14}$$

and using (12) and that $c \leq \bar{c}$, the following inequality follows:

$$\|y_k\| \leq \bar{c} r^k \|y_0\| + \sum_{i=0}^{k-1} \bar{c} r^{k-i-1} \varepsilon \|y_i\| \qquad \forall k \in \mathbb{N}^\star \tag{15}$$

Relation (15) can be written as

$$r^{-k} \|y_k\| \leq \bar{c} \|y_0\| + \sum_{i=0}^{k-1} \bar{c} r^{-1} \varepsilon (r^{-i} \|y_i\|) \qquad \forall k \in \mathbb{N}^\star \tag{16}$$

Gronwall's inequality for the discrete case provides

$$r^{-k} \|y_k\| \leq \bar{c} \|y_0\| \prod_{i=0}^{k-1} (1 + \bar{c} r^{-1} \varepsilon) \qquad \forall k \in \mathbb{N}^\star \tag{17}$$

thus

$$\|y_k\| \leq r^k \bar{c}\|y_0\|(1+\bar{c}r^{-1}\varepsilon)^k = \bar{c}\|y_0\|(r+\bar{c}\varepsilon)^k = \bar{c}\|y_0\|(\frac{r+1}{2})^k \qquad \forall k \in \mathbb{N}^\star \quad (18)$$

Denoting $\frac{r+1}{2} = \alpha < 1$, the relation (18) gives

$$\|x_k\| \leq \bar{c}\|x_{k_x}\|\alpha^k \qquad \forall k \geq k_x \tag{19}$$

Thus, for any $x \in DA(0)$ there exists $k_x \geq 0$ such that

$$\|f^k(x)\| \leq \bar{c}\|f^{k_x}(x)\|\alpha^k \qquad \forall k \geq k_x \tag{20}$$

which assures that the series $\sum\limits_{k=0}^{\infty} \|f^k(x)\|^2$ is convergent for any $x \in DA(0)$.

Let be $V = V(x)$ the function defined by

$$V(x) = \sum_{k=0}^{\infty} \|f^k(x)\|^2 \qquad \forall x \in DA(0) \tag{21}$$

The above function defined on $DA(0)$ is analytical, positive and satisfies (9). In order to show that the function V defined by (21) is the unique function which satisfies (9) we consider $V' = V'(x)$ satisfying (9) and we denote by V'' the difference $V'' = V - V'$. It is easy to see that $V''(f(x)) - V''(x) = 0$, for any $x \in DA(0)$. Therefore, we have $V''(x) = V''(f^k(x))$ for any $x \in DA(0)$ and any $k \in \mathbb{N}$. It follows that $V''(x) = \lim\limits_{k\to\infty} V''(f^k(x)) = 0$ for any $x \in DA(0)$. In other words, $V(x) = V'(x)$, for any $x \in DA(0)$, so V defined by (21) is the unique function which satisfies (9).

In order to show that $V(x) \overset{x\to x^0}{\longrightarrow} \infty$ for any $x^0 \in FrDA(0)$ we consider $x^0 \in FrDA(0)$ and $r > 0$ such that $\|f^k(x^0)\| > r$, for any $k = 0, 1, 2....$ For an arbitrary positive number $N > 0$ we consider the first natural number k_1 which satisfies $k_1 \geq \frac{2N}{r^2} + 1$. Let be $r_1 > 0$ such that $\|f^k(x)\| \geq \frac{r}{\sqrt{2}}$ for any $k = 1, 2, .., k_1$ and $x \in B(x^0, r_1)$. For any $x \in B(x^0, r_1) \cap DA(0)$ we have $\sum\limits_{k=0}^{k_1} \|f^k(x)\|^2 > N$. Therefore, $V(x) \overset{x\to x^0}{\longrightarrow} \infty$. □

3 Numerical examples

In this section, some examples of discrete systems are given, for which the domain of attraction of the zero steady state can be estimated using the method described in [2].

Example 1

The following system of difference equations is considered:

$$\begin{cases} x_{n+1} = x_n y_n + y_n \\ y_{n+1} = y_n^3 \end{cases} \tag{22}$$

It is clear that $(0,0)$ is an asymptotically stable fixed point for the system (22). It can be proved theoretically that the domain of attraction of $(0,0)$ is $DA(0) = \mathbb{R} \times (-1,1)$.
The matrix of the linearized system has the norm $\|\partial_0 f\| = 1$, thus the theorem from [2, 3] does not apply. On the other hand, $r(\partial_0 f) = 0$, thus the characterization theorem for $DA(0)$ given above applies. Thus, the domain of attraction can be estimated using the numerical method given in [2]. The first step of the method gives the whole domain of attraction:

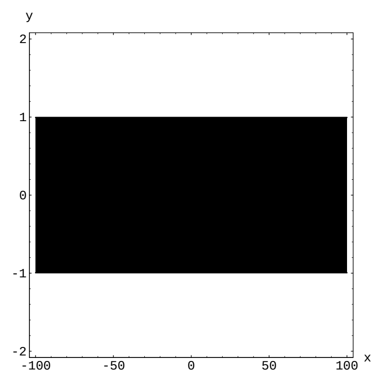

Figure 1: The estimate of $DA(0)$ after 1 step for system (22)

Example 2

The following system is considered:

$$\begin{cases} x_{n+1} = 4x_n^2 + y_n \\ y_{n+1} = x_n y_n \end{cases} \tag{23}$$

This system of difference equations has the asymptotically stable fixed point $(0,0)$. It is once again the case when $\|\partial_0 f\| = 1$ and $r(\partial_0 f) = 0$. After one step, we can obtain the following estimate of the $DA(0)$:

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Figure 2: The estimate of $DA(0)$ after 1 step for system (23)

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